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LETTER TO THE EDITOR

Disorder solutions and the star–triangle relation

Héctor J Giacomini

Instituto de Física de Rosario, IFIR, CONICET, UNR, Pellegrini 250, 2000 Rosario, Argentina

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Abstract. It is shown that the disorder solutions exhibited by two-dimensional statistical lattice systems verify the star-triangle relation. These solutions of the star-triangle relation are of a different type than those corresponding to completely integrable models.

The star-triangle relation (STR) has been shown to be a crucial element in the study of exactly solved two-dimensional models in statistical mechanics (Baxter 1980, 1982). The underlying reason is that the local STR is a sufficient condition for the commutation of global transfer matrices, which is an essential step in the resolution of the model.

The problem of solving the general STR is extremely difficult. The great number of unknowns and equations makes it practically impossible to solve them in the general case. Only reduced solutions corresponding to particular symmetries or restrictions, like symmetric vertex models, the hard-hexagon model (Baxter 1982), the restricted SOS model (Andrews *et al* 1984), etc, are known (see also Jimbo and Miwa 1985).

Very recently (Lochak and Maillard 1985) it has been shown that under rather mild restrictions the STR represents not only a sufficient condition but a necessary condition for the existence of commuting transfer matrices. This result enhances the prominent role played by the STR in exact solubility.

However, the studies carried out on the STR seem to show that this is a very restrictive condition, with very few non-trivial solutions. As has been stressed by many authors (e.g. Lochak and Maillard 1985) it would be desirable to extend the notion of integrability beyond it and to introduce new local criteria. In this direction, a typical example that has been given is the so-called disorder (or crystal growth) solutions. A great variety of anisotropic models (with different coupling constants in the different directions) are known to possess remarkable submanifolds in the space of parameters, where the partition function is computable and takes a very simple form. These are the disorder solutions, which have been obtained by using very different techniques: methods related to crystal growth (Enting 1978, Welberry and Miller 1978), to Markov processes (Verhagen 1976, Rujan 1984), to transfer matrices (Rujan 1982, Baxter 1984) and a procedure based on an exact decimation technique (Jaekel and Maillard 1985, Wu 1985).

Completely integrable models present disorder solutions, e.g. the triangular Ising model (Stephenson 1970) and the symmetric 8-vertex model (Baxter 1982). But very important models that are not integrable also present this type of solution, e.g. the triangular Ising model with a field (Verhagen 1976), the triangular q-state Potts model (Rujan 1984) and the general 8-vertex model (Peschel and Rys 1982, Rujan 1982, Giacomini 1986). The STR has not been used at all for obtaining these results, and therefore the idea exists in the literature that the disorder solutions do not have any relation to the STR. However in this letter it is shown that the STR is a sufficient condition for the existence of disorder solutions.

First we consider the case of spin systems. A very general model of this type is the IRF model on the square lattice (Baxter 1980). With each site i we associate a spin σ_i which can take any prescribed set of values (e.g. +1 or −1; or the integers $1, \ldots, q$). To each face we assign a Boltzmann weight factor $W(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, where $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are four spins round the face, arranged anticlockwise as in figure 1.

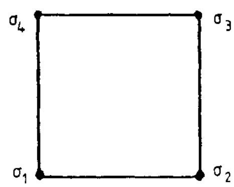

Figure 1. The four spins σ_1, σ_2, σ_3 and σ_4 round a face of the square lattice.

The partition function is

$$Z = \sum_{(\sigma)} \prod_{f} W(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$$

where the sum is over all values of all the spins and the product is over all faces of a square lattice with N^2 sites and with periodic boundary conditions.

For this model, the STR is as follows (Baxter 1982, Maillard and Garel 1984):

$$\sum_{\sigma} W(\sigma_1, \sigma_2, \sigma_3, \sigma) W'(\sigma_6, \sigma_1, \sigma, \sigma_5) W''(\sigma, \sigma_3, \sigma_4, \sigma_5)$$
$$= \sum_{\sigma'} W(\sigma_6, \sigma', \sigma_4, \sigma_5) W'(\sigma', \sigma_2, \sigma_3, \sigma_4) W''(\sigma_1, \sigma_2, \sigma', \sigma_6) \tag{1}$$

where $\sigma_1, \ldots, \sigma_6$ are fixed spins and the summation is over (σ, σ'). Here W, W' and W'' are the Boltzmann weights corresponding to three different choices of parameters of a given model.

If the STR (1) is satisfied the row-to-row transfer matrices $T_N(W)$ and $T_N(W')$ associated with the weights W and W' commute, even for arbitrary size N (Baxter 1980, Kasteleyn 1975). If one tries to solve equation (1) by eliminating the parameters with W'', one obtains a vanishing determinant condition like

$$\det(\mathrm{matrix}(W, W')) = 0 \tag{2}$$

because of the linear homogeneous character of (1). For exactly soluble models the equation (2) leads to relations of the form (Jaekel and Maillard 1983)

$$\varphi_{i,N}(W) = \varphi_{i,N}(W') \tag{3}$$

where $\varphi_{i,N}$ is an algebraic function of the parameters of the model (the index i indicates that there may be several functions for a given N). There always exist trivial solutions of (1), such as $W = \mathrm{constant} \times W'$ (which corresponds merely to the fact that T_N commutes with itself), and cases where the STR is satisfied for any weights W, W', W'', which often correspond to one- or zero-dimensional models in disguise.

In this letter we consider a new class of solutions of (1). They have the simple form

$$W' = W'' = 1 \tag{4}$$

for all spin configurations, and

$$\sum_{\sigma} W(\sigma_1, \sigma_2, \sigma_3, \sigma) = \sum_{\sigma'} W(\sigma_6, \sigma', \sigma_4, \sigma_5) \tag{5}$$

for all values of $\sigma_1, \ldots, \sigma_6$. It is assumed that the Boltzmann weight W satisfies the symmetry relation

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = W(\sigma_3, \sigma_4, \sigma_1, \sigma_2) \tag{6}$$

which corresponds to an invariance under a 180° rotation.

Now, using (6), equation (5) becomes

$$\sum_{\sigma} W(\sigma_1, \sigma_2, \sigma_3, \sigma) = \lambda \tag{7}$$

where λ is a constant independent of the spins $\sigma_1, \sigma_2, \sigma_3$. Hence, we obtain a solution of the STR (1) if the weights W, W' and W'' satisfy the equations (4), (6) and (7). In consequence, the following commutation relation holds:

$$[T_N(W), T_N(W' = 1)] = 0 \tag{8}$$

from which it follows that both matrices have a common set of eigenvectors. But all the elements of the matrix $T_N(W' = 1)$ are equal to one, and therefore their eigenvectors are of the form

$$X_1 = (1, \ldots, 1) \tag{9a}$$

and

$$X_i = (\alpha_i^{(1)}, \ldots, \alpha_i^{(2^N)}) \tag{9b}$$

with $i = 2, 3, \ldots, 2^N$ and $\Sigma_{j=1}^{2^N} \alpha_i^{(j)} = 0$.

In the thermodynamic limit, the partition function per site κ is given by

$$\kappa = \lim_{N\to\infty} Z^{1/N^2} = \Lambda^{1/N} \tag{10}$$

where Λ is the maximum eigenvalue of the transfer matrix. But, for physical non-negative Boltzmann weights, the maximum eigenvalue corresponds to the eigenvector with all its elements non-negative, i.e. the eigenvector (9a).

From (7) and (9a), and the expression of $T_N(W)$ in terms of the weights W (Baxter 1982), it is straightforward to see that

$$\Lambda^{1/N} = \sum_{\sigma} W(\sigma_1, \sigma_2, \sigma_3, \sigma) = \lambda. \tag{11}$$

In this way, we have obtained the exact solution of the IRF model when (6) and (7) are satisfied, from the commutation property (8), which, in turn, was derived from the solution of the STR (1).

Now it will be shown that this solution is indeed a disorder solution. In fact, Baxter (1984) has given a sufficient condition for an IRF model on a square lattice to have a disorder solution. It is as follows: if there exists a parameter κ and a single-spin function $\varphi(\sigma)$ such that

$$\sum_{\sigma_4} W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) g(\sigma_1, \sigma_4) f(\sigma_3, \sigma_4) = \kappa \varphi(\sigma_1) f(\sigma_1, \sigma_2) g(\sigma_2, \sigma_3) / \varphi(\sigma_3) \tag{12}$$

for all values of σ_1, σ_2, σ_3, where $f(a, b)$ and $g(a, b)$ are non-negative functions, then the model is at a disorder point and κ is the partition function per site. Moreover, if the additional condition

$$\sum_{\sigma_2} W(\sigma_1, \sigma_2, \sigma_3, \sigma_4)\tilde{f}(\sigma_1, \sigma_2)\tilde{g}(\sigma_2, \sigma_3) = \kappa\tilde{\varphi}(\sigma_1)\tilde{g}(\sigma_1, \sigma_4)\tilde{f}(\sigma_3, \sigma_4)/\tilde{\varphi}(\sigma_3) \tag{13}$$

where $\tilde{f}$ and $\tilde{g}$ are non-negative functions, is satisfied, then the intra-row correlations have a one-dimensional behaviour, and therefore there can be no long-range order.

It is evident that if the Boltzmann weight W satisfies the condition (6), equations (12) and (13) are equivalent if we take $\tilde{f} = f$ and $\tilde{g} = g$. Now, due to the periodic boundary conditions, the partition function is left unchanged by the transformations (Maillard and Garel 1984)

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \to \frac{D_1(\sigma_1, \sigma_4)}{D_1(\sigma_2, \sigma_3)} W(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$$

and (14)

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \to \frac{D_2(\sigma_1, \sigma_2)}{D_2(\sigma_3, \sigma_4)} W(\sigma_1, \sigma_2, \sigma_3, \sigma_4).$$

The positions of the σ_i are shown in figure 1. Making use of this property, Baxter's condition (12) can be written as

$$\sum_{\sigma_2} W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \kappa$$

which is simply equation (11), resulting from the solution of the STR. Therefore we see that the special type of solution (4) and (7) of the STR leads to a disorder solution of the model in question.

For vertex models on a square lattice we obtain similar results. In this case the STR takes the form (Baxter 1985)

$$\sum_{\lambda,\mu,\nu} S\begin{pmatrix}\alpha & \lambda\\ \beta & \mu\end{pmatrix} S'\begin{pmatrix}\varepsilon & \nu\\ \phi & \lambda\end{pmatrix} S''\begin{pmatrix}\gamma & \mu\\ \delta & \nu\end{pmatrix} = \sum_{\lambda,\mu,\nu} S''\begin{pmatrix}\nu & \alpha\\ \mu & \phi\end{pmatrix} S'\begin{pmatrix}\lambda & \gamma\\ \nu & \beta\end{pmatrix} S\begin{pmatrix}\mu & \varepsilon\\ \lambda & \delta\end{pmatrix} \tag{15}$$

where $S\left(\begin{smallmatrix}\alpha & \lambda\\ \beta & \mu\end{smallmatrix}\right)$ is the Boltzmann weight associated with the four links round a vertex, as shown in figure 2. Analogously to (4) we take

$$S' = S'' = 1 \tag{16}$$

for all the configurations of links.

Provided the symmetry condition

$$S\begin{pmatrix}\alpha & \lambda\\ \beta & \mu\end{pmatrix} = S\begin{pmatrix}\mu & \beta\\ \lambda & \alpha\end{pmatrix} \tag{17}$$

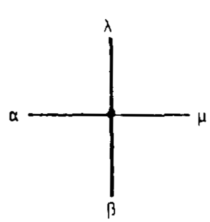

Figure 2. The four links α, λ, β and μ round a vertex of the square lattice.

is satisfied, equation (15) becomes

$$\sum_{\lambda,\mu} S\begin{pmatrix}\alpha & \lambda \\ \beta & \mu\end{pmatrix} = \kappa \tag{18}$$

where κ, as above, is the partition function per site. This equation is the analogue of (11), valid for spin systems. All the calculations for vertex models are similar to the IRF case.

Now, vertex models on a square lattice can be expressed as spin systems on a checkerboard lattice (Suzuki and Fisher 1971). Using this equivalence, the condition (18) is identical to the local criterion for obtaining disorder solutions of spin models on checkerboard lattices, introduced by Jaekel and Maillard (1985). Therefore, we have shown that disorder solutions for vertex and spin models can be obtained from an adequate type of solution of the STR.

All the disorder solutions that have been obtained so far are valid only in the thermodynamic limit and for real non-negative values of the Boltzmann weights. This is unfortunate since the knowledge of the disorder solutions for complex values of the parameters enables us to analyse important questions about the partition function (zeros, singularities, etc).

The results of this letter help to enforce the idea that the STR is a crucial element for obtaining exact results of two-dimensional statistical lattice models and the prospect of obtaining weaker conditions than the STR for exact solubility decreases.

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